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SENIOR SCHOOL

AS/A Level Further Mathematics

AS/A level Further Mathematics course structure

Further Mathematics is taken **in addition** to A level Mathematics

AS level

Pure Mathematics	Additional Pure, Statistics, Mechanics or Decision Mathematics

- Pure mathematics content makes up 50% of the AS level and 50% of the A level.

A level

Pure Mathematics	Additional Pure, Statistics, Mechanics or Decision Mathematics

FPSS entry requirements

AS/A level Further Mathematics

- At least **grade 7** in GCSE Maths.

Taster lessons

- Lesson 1 – Series (Core Pure 1)
- Lesson 2 – Complex Numbers (Core Pure 1)

SERIES

Lesson 1

DO NOW:

Simplify each expression by writing it as the product of two factors:

a $(k + 1) + (k + 1)(k + 2)$

b $\frac{1}{2}(k + 1)^2 + k^2(k + 1)^2$

c $k^2(2k - 1) + 10k - 5$

a $(k + 1)(1 + k + 2) = (k + 1)(k + 3)$

b $\frac{1}{2}(k + 1)^2(1 + 2k^2)$

c $(2k - 1)(k^2 + 5)$

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Sigma notation

A **series** is a **sum** of (a finite or infinite number of) terms.

The Σ symbol ("capital sigma") indicates a summation.

Determine the following results by explicitly writing out the elements in the sum:

$\Sigma_{p=3}^{8} p^2 =$?

$\Sigma_{r=0}^{5} (7r+1)^2 =$?

Sum of 1s and natural numbers

$$\sum_{r=1}^{n} 1 = \boxed{?}$$

Sum of first n natural numbers

$$\sum_{r=1}^{n} r = \boxed{?}$$

Quick fire: triangulars!

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

1 + 2 + 3 + … + 10 = ?

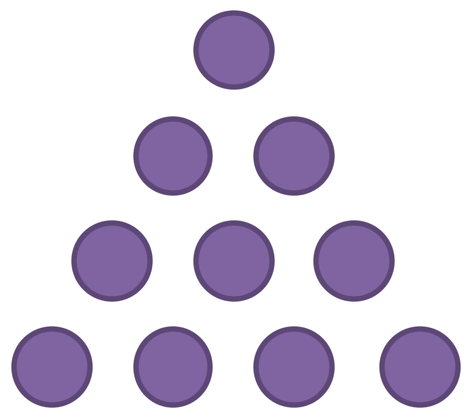
1 + 2 + 3 + … + 99 = ?

11 + 12 + 13 + … + 20 = ?

Sum up to 20, but get rid of everything up to 10:

$$\left(\frac{1}{2} \times 20 \times 21\right) - \left(\frac{1}{2} \times 10 \times 11\right)$$

100 + 101 + 102 + … + 200 = ?



Evaluate…

$$\sum_{r=n}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{4}(2r-1) = \;?$$

There are sufficiently few terms that we can just list them.

$$\sum_{r=1}^{50} r = \;?$$

$$\sum_{r=25}^{50} r = \;?$$

Tip: For summations where you subtract, ensure that you use **one less** than the lower limit.

Show that $\sum_{r=5}^{2N-1} r = 2N^2 - N - 10$ (for $N \geq 3$)

?

We substitute n for whatever the upper limit is, in this case, $2N - 1$

Test your understanding...

Show that $\sum_{r=n}^{3n} r = 2n(2n+1)$

Tip: After writing out your initial subtraction, **DO NOT expand out** – is there a common term you can factorise?

?

?

Breaking up summations

Prove that $\sum_{r=1}^{n} kr = k\sum_{r=1}^{n} r$, where k is a constant.

?

Examples:

$$\sum_{r=1}^{n} 3r =$$? = ?

$$\sum_{r=1}^{n} 4 =$$? = ?

Breaking up summations

Prove that $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$$\sum_{i=1}^{n}(a_i + b_i) = \text{?}$$

$$= \text{?}$$

$$= \text{?}$$

We can combine this property of summations with the previous one to break summations up:

$$\sum_{r=1}^{25}(3r + 1) = \text{?}$$

$$= \text{?}$$

Test your understanding

Show that $\sum_{r=1}^{n}(3r+2)=\frac{n}{2}(3n+7)$

?

Hence evaluate $\sum_{r=20}^{50}(3r+2)$

?

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Your turn...

1. Find

   (a) $\sum_{r=30}^{50}\left(r^3-2\right)$ (b) $\sum_{r=10}^{20}r\left(r-2\right)$

2. Using the standard results, find an expression for $\sum_{r=1}^{n}(r+1)(2r+1)$ and hence evaluate $2\times3+3\times5+4\times7+\ldots\ldots\ldots+21\times41$.

3. Using the standard results, find the sum of the first n terms of $\left(r+4\right)^3$.

4. Write down an expression for the rth term of the series

   $$1\times2+3\times4+5\times6+\ldots\ldots$$

   and hence find the sum of the first n terms.

5. Using the standard result for $\sum_{1}^{n}r^3$, show that $\sum_{r=n+1}^{2n}r^3=\dfrac{n^2(3n+1)(5n+3)}{4}$.

Answers

1. (a) $\sum_{r=30}^{50}(r^3-2)=\sum_{r=1}^{50}(r^3-2)-\sum_{r=1}^{29}(r^3-2)$

$$\sum_{r=1}^{50}(r^3-2)=\sum_{r=1}^{50}r^3-\sum_{r=1}^{50}2$$
$$=\left(\tfrac{1}{4}\times50^2\times51^2\right)-\left(50\times2\right)$$
$$=1625625-100$$
$$=1625525$$
$$\sum_{r=1}^{29}(r^3-2)=\sum_{r=1}^{29}r^3-\sum_{r=1}^{29}2$$
$$=\left(\tfrac{1}{4}\times29^2\times30^2\right)-\left(29\times2\right)$$
$$=189225-58$$
$$=189167$$
$$\sum_{r=30}^{50}(r^3-2)=1625525-189167=1436358$$

(b) $\sum_{r=10}^{20}r(r-2)=\sum_{r=1}^{20}r(r-2)-\sum_{r=1}^{9}r(r-2)$

$$\sum_{r=1}^{20}r^2-2r=\sum_{r=1}^{20}r^2-2\sum_{r=1}^{20}r$$
$$=\left(\tfrac{1}{6}\times20\times21\times41\right)-2\left(\tfrac{1}{2}\times20\times21\right)$$
$$=2870-420$$
$$=2450$$
$$\sum_{r=1}^{9}r^2-2r=\sum_{r=1}^{9}r^2-2\sum_{r=1}^{9}r$$
$$=\left(\tfrac{1}{6}\times9\times10\times19\right)-2\left(\tfrac{1}{2}\times9\times10\right)$$
$$=285-90$$
$$=195$$
$$\sum_{r=10}^{20}r(r-2)=2450-195=2255$$

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Answers

2. $$\sum_{r=1}^{n}(r+1)(2r+1)=\sum_{r=1}^{n}(2r^2+3r+1)$$

$$=2\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$

$$=2\times\tfrac{1}{6}n(n+1)(2n+1)+3\times\tfrac{1}{2}n(n+1)+n$$

$$=\tfrac{1}{6}n\left(4n^2+6n+2+9n+9+6\right)$$

$$=\tfrac{1}{6}n(4n^2+15n+17)$$

Putting n = 20:

$2\times3+3\times5+4\times7+\ldots+21\times41$

$=\tfrac{1}{6}\times20(4\times20^2+15\times20+17)$

$=6390$

3. $$\sum_{r=1}^{n}(r+4)^3=\sum_{r=1}^{n+4}r^3-\sum_{r=1}^{r=4}r^3$$

$$=\tfrac{1}{4}(n+4)^2(n+5)^2-\tfrac{1}{4}\times4^2\times5^2$$

$$=\tfrac{1}{4}\left((n^2+9n+20)^2-400\right)$$

$$=\tfrac{1}{4}\left(n^4+18n^3+121n^2+360n+400-400\right)$$

$$=\tfrac{1}{4}n(n^3+18n^2+121n+360)$$

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Answers

4. The rth term is $2r\left(2r-1\right)$.

$$\sum_{r=1}^{n}\left(2r(2r-1)\right)=4\sum_{r=1}^{n}r^2-2\sum_{r=1}^{n}r$$
$$=4\times\tfrac{1}{6}n(n+1)(2n+1)-2\times\tfrac{1}{2}n(n+1)$$
$$=\tfrac{1}{3}n(n+1)\left(4n+2-3\right)$$
$$=\tfrac{1}{3}n(n+1)(4n-1)$$

5. $\sum_{1}^{n}r^3=\tfrac{1}{4}n^2(n+1)^2$

$$\sum_{r=n+1}^{2n}r^3=\sum_{r=1}^{2n}r^3-\sum_{r=1}^{n}r^3$$
$$=\tfrac{1}{4}(2n)^2(2n+1)^2-\tfrac{1}{4}n^2(n+1)^2$$
$$=\tfrac{1}{4}n^2\left(4(2n+1)^2-(n+1)^2\right)$$
$$=\tfrac{1}{4}n^2(16n^2+16n+4-n^2-2n-1)$$
$$=\tfrac{1}{4}n^2(15n^2+14n+3)$$
$$=\tfrac{1}{4}n^2(3n+1)(5n+3)$$

Any questions?

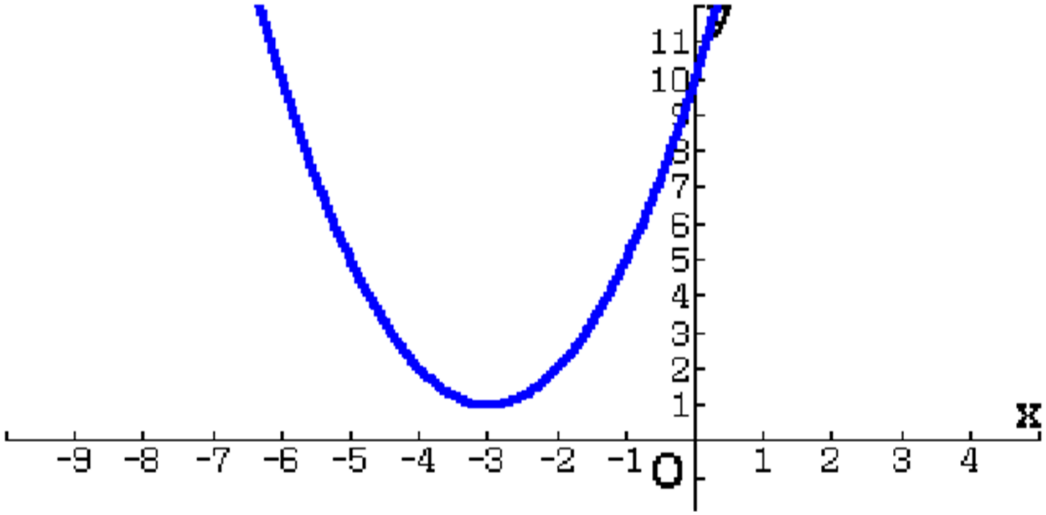
COMPLEX NUMBERS

Lesson 2

DO NOW:

1. Sketch a graph of $y = x^2 + 5x - 9$, labelling any points of intersection with the co-ordinate axes and the minimum of the curve.

2. Explain algebraically why the equation $x^2 + 6x + 10 = 0$ has no real solutions.



3. Letting $i = \sqrt{-1}$ (pretending that it exists), express the solutions of $x^2 + 6x + 10 = 0$ in terms of i.

Why complex numbers?

Remember when you used the quadratic formula, and "it broke" when you had a negative discriminant? (the $b^2 - 4ac$ bit)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By adding $i = \sqrt{-1}$ to our number system, we can then represent roots of quadratics that we couldn't previously do so with "real" numbers (i.e. numbers in $\mathbb{R}$).

- $i = \sqrt{-1}$
- An **imaginary** number is of the form bi where $b \in \mathbb{R}$, e.g. $i, 3i, -2i, i\pi$
- A **complex** number is of the form $a + bi$, where $a, b \in \mathbb{R}$, e.g. $1 + i, 3 - 2i$
- We say that a is the "real part" and b the "imaginary part" of the number.

Video clips

Imaginary Numbers Are Real [Part 1: Introduction]

Imaginary Numbers Are Real [Part 2: A Little History]

Imaginary Numbers Are Real [Part 3: Cardan's Problem]

Imaginary Numbers Are Real [Part 4: Bombelli's Solution]

Imaginary Numbers Are Real [Part 5: Numbers are Two Dimensional]

The basics

Write the following in terms of i:

$\sqrt{-36} =$?

$\sqrt{-4} =$?

$\sqrt{-7} =$?

Simplify:

$(2+3i)+(4+i) =$?

$i-3(2-i) =$?

$\dfrac{10+4i}{2} =$?

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Your turn...

1 Write each of the following in the form bi where b is a real number.

a $\sqrt{-9}$ **b** $\sqrt{-49}$ **c** $\sqrt{-121}$ **d** $\sqrt{-10000}$ **e** $\sqrt{-225}$

f $\sqrt{-5}$ **g** $\sqrt{-12}$ **h** $\sqrt{-45}$ **i** $\sqrt{-200}$ **j** $\sqrt{-147}$

2 Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

a $(5 + 2i) + (8 + 9i)$ **b** $(4 + 10i) + (1 - 8i)$

c $(7 + 6i) + (-3 - 5i)$ **d** $\left(\frac{1}{2} + \frac{1}{3}i\right) + \left(\frac{5}{2} + \frac{5}{3}i\right)$

e $(20 + 12i) - (11 + 3i)$ **f** $(2 - i) - (-5 + 3i)$

g $(-4 - 6i) - (-8 - 8i)$ **h** $(3\sqrt{2} + i) - (\sqrt{2} - i)$

i $(-2 - 7i) + (1 + 3i) - (-12 + i)$ **j** $(18 + 5i) - (15 - 2i) - (3 + 7i)$

3 Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

a $2(7 + 2i)$ **b** $3(8 - 4i)$

c $2(3 + i) + 3(2 + i)$ **d** $5(4 + 3i) - 4(-1 + 2i)$

e $\frac{6 - 4i}{2}$ **f** $\frac{15 + 25i}{5}$

g $\frac{9 + 11i}{3}$ **h** $\frac{-8 + 3i}{4} - \frac{7 - 2i}{2}$

Answers

1

a $\sqrt{9}\sqrt{(-1)} = 3i$

b $\sqrt{49}\sqrt{(-1)} = 7i$

c $\sqrt{121}\sqrt{(-1)} = 11i$

d $\sqrt{1000}\sqrt{(-1)} = 100i$

e $\sqrt{225}\sqrt{(-1)} = 15i$

f $\sqrt{5}\sqrt{(-1)} = i\sqrt{5}$

g $\sqrt{12}\sqrt{(-1)} = \sqrt{4}\sqrt{3}\sqrt{(-1)} = 2i\sqrt{3}$

h $\sqrt{45}\sqrt{(-1)} = \sqrt{9}\sqrt{5}\sqrt{(-1)} = 3i\sqrt{5}$

i $\sqrt{200}\sqrt{(-1)} = \sqrt{100}\sqrt{2}\sqrt{(-1)} = 10i\sqrt{2}$

j $\sqrt{147}\sqrt{(-1)} = \sqrt{49}\sqrt{3}\sqrt{(-1)} = 7i\sqrt{3}$

2

a $(5+8)+i(2+9) = 13+11i$

b $(4+1)+i(10-8) = 5+2i$

c $(7-3)+i(6-5) = 4+i$

d $\left(\frac{1}{2}+\frac{5}{2}\right)+i\left(\frac{1}{3}+\frac{5}{3}\right) = 3+2i$

e $(20-11)+i(12-3) = 9+9i$

f $(2--5)+i(-1-3) = 7-4i$

g $(-4--8)+i(-6--8) = 4+2i$

h $(3\sqrt{2}-\sqrt{2})+i(1--1) = 2\sqrt{2}+2i$

i $(-2+1--12)+i(-7+3-1) = 11-5i$

j $(18-15-3)+i(5--2-7) = 0$

3

a $14+4i$

b $24-12i$

c $(6+2i)+(6+3i) = (6+6)+i(2+3)$
$= 12+5i$

d $(20+15i)+(4-8i) = (20+4)+i(15-8)$
$= 24+7i$

e $\frac{6-4i}{2} = \frac{6}{2}-\frac{4}{2}i$
$= 3-2i$

f $\frac{15+25i}{5} = \frac{15}{5}-\frac{25}{5}i$
$= 3+5i$

g $\frac{9+11i}{3} = \frac{9}{3}+\frac{11}{3}i$
$= 3+\frac{11}{3}i$

h $\frac{-8+3i}{4}-\frac{7-2i}{2} = -2+\frac{3}{4}i-\frac{7}{2}+i$
$= \left(-\frac{4}{2}-\frac{7}{2}\right)+i\left(\frac{3}{4}+\frac{4}{4}\right)$
$= -\frac{11}{2}+\frac{7}{4}i$

Multiplying complex numbers

Given that $i = \sqrt{-1}$, it follows that $\boldsymbol{i^2 = -1}$

Determine the value of i^3, i^4, i^{101} and $(3i)^5$

Express each of the following in the form $a + bi$, where a, b are integers.

1) $(2+3i)(3-2i)$
2) $(5-3i)^2$

i^3
$=$?
$=$

i^4
$=$?
$=$

$(2+3i)(3-2i)$
$=$?
$=$
$=$

We can therefore see that for increasing powers of i, we obtain $i, -1, -i, 1$ where $i^k = 1$ if k is a multiple of 4.

i^{101}
$=$?
$=$

$(5-3i)^2$
$=$?
$=$
$=$
$=$

$(3i)^5 =$?

Complex roots

You can use complex numbers to find solutions to any quadratic equation with real coefficients.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which are real.**

1 Solve each of the following equations. Write your answers in the form $\pm b\text{i}$.

a $z^2 + 121 = 0$ **b** $z^2 + 40 = 0$ **c** $2z^2 + 120 = 0$

d $3z^2 + 150 = 38 - z^2$ **e** $z^2 + 30 = -3z^2 - 66$ **f** $6z^2 + 1 = 2z^2$

2 Solve each of the following equations. Write your answers in the form $a \pm b\text{i}$.

a $(z - 3)^2 - 9 = -16$

b $2(z - 7)^2 + 30 = 6$

c $16(z + 1)^2 + 11 = 2$

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Answers

1 a $z^2+121=0$

$z^2=-121$

$z=\pm\sqrt{-121}$

$=\pm\sqrt{121}\sqrt{-1}$

$=\pm 11\text{i}$

b $z^2+40=0$

$z^2=-40$

$z=\pm\sqrt{-40}$

$=\pm\sqrt{4}\sqrt{10}\sqrt{-1}$

$=\pm\sqrt{4}\sqrt{10}\text{i}$

$=\pm 2\text{i}\sqrt{10}$

c $2z^2+120=0$

$2z^2=-120$

$z^2=-60$

$z=\pm\sqrt{-60}$

$=\pm\sqrt{4}\sqrt{15}\sqrt{-1}$

$=\pm 2\text{i}\sqrt{15}$

d $3z^2+150=38-z^2$

$4z^2=-112$

$z^2=-28$

$z=\pm\sqrt{-28}$

$=\pm\sqrt{4}\sqrt{7}\sqrt{-1}$

$=\pm 2\text{i}\sqrt{7}$

e $z^2+30=-3z^2-66$

$4z^2=-96$

$z^2=-24$

$z=\pm\sqrt{-24}$

$=\pm\sqrt{4}\sqrt{6}\sqrt{-1}$

$=\pm 2\text{i}\sqrt{6}$

1 f

$6z^2+1=2z^2$

$4z^2=-1$

$z^2=-\frac{1}{4}$

$z=\pm\sqrt{\frac{-1}{4}}$

$=\pm\sqrt{\frac{1}{4}}\sqrt{-1}$

$=\pm\frac{1}{2}\text{i}$

2 a $(z-3)^2-9=-16$

$(z-3)^2=-7$

$z-3=\pm\text{i}\sqrt{7}$

$z=3\pm\text{i}\sqrt{7}$

b $2(z-7)^2+30=6$

$2(z-7)^2=-24$

$(z-7)^2=-12$

$z-7=\pm 2\text{i}\sqrt{3}$

$z=7\pm 2\text{i}\sqrt{3}$

c $16(z+1)^2+11=2$

$16(z+1)^2=-9$

$(z+1)^2=-\frac{9}{16}$

$z+1=\pm\frac{3}{4}\text{i}$

$z=-1\pm\frac{3}{4}\text{i}$

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Any questions?